25025033

QP CODE: 25025033

Reg No : .....

Name : .....

M.Sc DEGREE (CSS) EXAMINATION, MAY 2025

Second Semester

M Sc MATHEMATICS

CORE - ME010201 - ADVANCED ABSTRACT ALGEBRA

2019 ADMISSION ONWARDS

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Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. *Define an algebraically closed field. Give an example.*
2. *If α and β are constructible real numbers, then prove that $\alpha+\beta$ is constructible.*
3. Let R be a commutative ring with unity. Define i) unit of R, ii) associate elements in R.
4. Prove that $\mathbb{Z}[i]$ is an integral domain.
5. Prove that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain.
6. *Let $E=\mathbb{Q}(\sqrt{2},\sqrt{3})$ and $\sigma: E \to E$ defined by $\sigma(a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}) = a+b\sqrt{2}-c\sqrt{3}-d\sqrt{6}$ for $a,b,c,d \in \mathbb{Q}$. Find the fixed field of σ.*
7. *State the isomorphism extension theorem. Use this theorem to prove that if $E \leq \overline{F}$ is an algebraic extension of a field F and $\alpha, \beta \in E$ are conjugate over F, then the conjugation isomorphism $\psi_{\alpha,\beta}: F(\alpha) \to F(\beta)$ can be extended to an isomorphism of E onto a subfield of $\overline{F}$.*
8. *Describe all extensions of the identity map of $\mathbb{Q}$ to an isomorphism mapping $\mathbb{Q}(\sqrt[3]{2})$ onto a subfield of $\overline{Q}$.*
9. *Let $f(x)$ be a polynomial in $F[x]$ where F is a field. Define the group of $f(x)$ over F.*
10. *Show that x^4+1 is irreducible in $\mathbb{Q}[x]$.*

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Find the degree and a basis for $\mathbb{Q}(\sqrt[3]{2},\sqrt{3})$ over $\mathbb{Q}$
12. *If F is a field of prime characteristic p, then prove that $(\alpha+\beta)^{p^n} = \alpha^{p^n}+\beta^{p^n}$ for all $\alpha, \beta \in F$ and all positive integers n.*
13. Prove that every PID is a UFD.

14. If F is a field and $x_1, x_2,...,x_n$ are indeterminates, then prove that $F[x_1, x_2,...,x_n]$ is a UFD.

15. *Let $f(x) \in \mathbb{R}[x]$. If $f(a+bi) = 0$, prove that $f(a-bi) = 0$.*

16. *Show that $\mathbb{Q}(\sqrt[3]{2})$ has only the identity automorphism.*

17. *If E is a finite extension of a field F, then prove that E is separable over F if and only if each α in E is separable over F.*

18. Let K be a finite normal extension of a field F and let E be an extension of F, where $F \leq E \leq K \leq \overline{F}$. Prove the following.
    a) K is a finite normal extension of E.
    b) $G(K/E)$ is precisely the subgroup of $G(K/F)$ consisting of all those automorphisms that leave E fixed.
    c) Two automorphisms σ and τ in $G(K/F)$ induce the same isomorphism of E onto a subfield of $\overline{F}$ if and only if they are in the same left coset of $G(K/E)$ in $G(K/F)$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Given E as an extension field of F and let $\alpha \in E$, where α is algebraic over F. Then prove
    a) There is an irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha)=0$.
    b) This irreducible polynomial p(x) is uniquely determined up to a constant factor in F and is a polynomial of minimal degree in F[x] having α as a zero.
    c) If $f(\alpha)=0$ for $f(x) \in F[x]$ with $f(x)\neq 0$, then $p(x)$ divides $f(x)$.

20. a) Define Euclidean domain
    b) Prove that every Euclidean domain is a PID.
    c) For a Euclidean domain with a Euclidean norm v, prove that v(1) is minimal among all v(a) for nonzero a in D, and also prove that u in D is a unit if and only if v(a) = v(1).

21. *Define splitting field over a field F. Prove that a field E, $F \leq E \leq \overline{F}$, is a splitting field over F if and only if every automorphism of $\overline{F}$ leaving F fixed induces an automorphism of E leaving F fixed.*

22. Prove the following.
    a) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Then all zeros of $f(x)$ in $\overline{F}$ have the same multiplicity.
    b) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Then $f(x)$ has a factorization in $\overline{F}[x]$ of the form $a\prod_i (x-\alpha_i)^{\nu}$ where α_i are the distinct zeros of $f(x)$ in $\overline{F}$ and $a \in F$.
    c) If E is a finite extension of a field F, then $\{E:F\}$ divides $[E:F]$.

(2×5=10 weightage)